

Integrable operators and squares of Hankel matrices

Andrew McCafferty

Department of Mathematics and Statistics, Lancaster University, Lancaster, LA1 4YF
(a.mccafferty@lancaster.ac.uk)

Abstract

In this note, we find sufficient conditions for an operator with kernel of the form $A(x)B(y) - A(x)B(y)/(x - y)$ (which we call a Tracy–Widom type operator) to be the square of a Hankel operator. We consider two contexts: infinite matrices on ℓ^2 , and integral operators on the Hardy space $H^2(\mathbb{T})$. The results can be applied to the discrete Bessel kernel, which is significant in random matrix theory.

Keywords: discrete-time Lyapunov equation, Tracy–Widom operator, Hankel operator

2000 Mathematics subject classification: 47B35, 15A52

1 Introduction

In random matrix theory it is natural (see, e.g. [1]) to consider integrable operators T , where the kernel of T is

$$\sum_{j=1}^n \frac{A_j(z)B_j(w)}{z - w}, \quad (1)$$

and $\sum_{j=1}^n A_j(z)B_j(z) = 0$. Here we are concerned with a special class of such operators, namely those with kernel of the form

$$K(x, y) = \frac{A(x)B(y) - A(y)B(x)}{x - y} \quad (x \neq y), \quad (2)$$

which we shall refer to as *Tracy–Widom* operators. The variables x and y may be non-negative integers, as in the discrete kernels considered in section 2, continuous real parameters, as in e.g. [2], or may live on the circle, as in section 3. We look for conditions under which these operators can be expressed as Γ^2 or $\Gamma^*\Gamma$, where Γ is a Hankel operator. In particular we recover a result of Borodin *et al* [3], showing that the discrete Bessel kernel can be written as

$$\sqrt{\theta} \frac{J_x(2\sqrt{\theta})J_{y+1}(2\sqrt{\theta}) - J_y(2\sqrt{\theta})J_{x+1}(2\sqrt{\theta})}{x - y} = \sum_{k=0}^{\infty} J_{x+k+1}(2\sqrt{\theta})J_{y+k+1}(2\sqrt{\theta}). \quad (3)$$

We can then read off information about K from knowledge of the Hankel operator Γ . For example, a trace formula follows immediately, and the spectrum of K can be calculated from the spectrum of Γ (which in many cases is easier to calculate). Megretski, Peller and Treil [4] have characterised the self-adjoint bounded linear operators that are unitarily equivalent to Hankel operators: we apply their results to gain spectral information about the operators K .

2 Discrete integrable operators

Define $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. We consider infinite matrices with kernel $K(x, y)$, where $K(x, y)$ is defined by (2). Recall that a Hankel matrix $\Gamma_\phi = [\phi(m+n)]_{m,n \geq 0}$ with $(\phi(k)) \in \ell^2$ has square

$$\Gamma_\phi^2 = \left[\sum_{k=0}^{\infty} \phi(m+k)\phi(n+k) \right]_{m,n=0}^{\infty}. \quad (4)$$

Nehari's theorem (see, e.g. [5, p. 3]) states that Γ_ϕ is a bounded operator on $\ell^2(\mathbb{N}_0)$ if and only if $(\phi(n))$ are the positive Fourier coefficients of some function in $L^\infty(\mathbb{T})$. We write the kernel $K(x, y)$ in matricial form,

$$\begin{aligned} K(x, y) &= \frac{1}{x-y} \langle F\mathbf{a}(x), \mathbf{a}(y) \rangle, \quad (x \neq y) \\ \mathbf{a}(x) &= \begin{bmatrix} A(x) \\ B(x) \end{bmatrix}, \quad F = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \end{aligned} \quad (5)$$

and look for sufficient conditions under which we can construct a function $\phi : \mathbb{N}_0 \rightarrow \mathbb{C}$ with $(\phi(j)) \in \ell^2$, such that

$$K(x, y) = \sum_{k=0}^{\infty} \phi(x+k)\phi(y+k), \quad (x \neq y). \quad (6)$$

Definition 2.1 Let S be the shift operator on $\ell^2(\mathbb{N}_0)$, so that $Sf(x) = f(x-1)$ (where we define $f(-1) = 0$), and let R be the adjoint shift operator $Rf(x) = f(x+1)$. The forward difference operator Δ is defined by $\Delta f(x) = f(x+1) - f(x)$. Notice that $\Delta = R_x - I$. Where there are several variables, we write R_x, Δ_y and so on.

As usual, A^T is the transpose of a matrix A , while B^* denotes the adjoint of an operator B .

Lemma 2.2 (Lyapunov equation) Suppose that R and B are bounded linear operators on ℓ^2 such that

$$\sum_{j=0}^{\infty} \langle R^j B B^* (R^*)^j \xi, \xi \rangle < \infty \quad \text{for all } \xi \in \ell^2,$$

so that the series

$$K = \sum_{j=0}^{\infty} R^j B B^* (R^*)^j$$

is convergent in the weak operator topology. Then

$$K - R K R^* = -B B^*. \quad (7)$$

Proof. Clear from calculation of the left hand side of (7). ■

In the following Lemma, we state explicitly the specialisation of the above result to discrete kernels.

Lemma 2.3 Let $\Phi(x, y)$ be any function $\Phi : \mathbb{N}_0^2 \rightarrow \mathbb{C}$, and suppose $\phi : \mathbb{N}_0 \rightarrow \mathbb{C}$ is such that $(\phi(j)) \in \ell^2$. Then

$$\Phi(x, y) = \sum_{k=0}^{\infty} \phi(x+k)\phi(y+k) \quad \text{for all } x, y \in \mathbb{N}_0 \quad (8)$$

if and only if

$$(\Delta_x S_y + \Delta_y) \Phi(x, y) = -\phi(x)\phi(y) \quad \text{for all } x, y \in \mathbb{N}_0 \quad (9)$$

and

$$\Phi(x, y) \rightarrow 0 \quad \text{as } x \text{ or } y \rightarrow \infty. \quad (10)$$

Proof. Suppose (8) holds. Then we have

$$\begin{aligned} (\Delta_x S_y + \Delta_y) \sum_{k=0}^{\infty} \phi(x+k)\phi(y+k) &= (S_x S_y - I) \sum_{k=0}^{\infty} \phi(x+k)\phi(y+k) \\ &= \sum_{k=0}^{\infty} (\phi(x+k+1)\phi(y+k+1) - \phi(x+k)\phi(y+k)) \\ &= -\phi(x)\phi(y), \end{aligned} \quad (11)$$

so that (9) holds. By the Cauchy-Schwarz inequality, and since $(\phi(j)) \in \ell^2$, we have

$$\begin{aligned} \Phi(x, y) &= \sum_{k=0}^{\infty} \phi(x+k)\phi(y+k) \\ &\leq \left(\sum_{k=0}^{\infty} \phi(x+k)^2 \right)^{1/2} \left(\sum_{k=0}^{\infty} \phi(y+k)^2 \right)^{1/2} \rightarrow 0 \quad \text{as } x \text{ or } y \rightarrow \infty, \end{aligned} \quad (12)$$

which is condition (10). Conversely, suppose that we have (9) and (10), and let

$$G(x, y) = \Phi(x, y) - \sum_{k=0}^{\infty} \phi(x+k)\phi(y+k). \quad (13)$$

By (9), we have

$$(\Delta_x S_y + \Delta_y) G(x, y) = 0 \quad \text{for all } x, y \in \mathbb{N}_0,$$

so that $G(x, y) = G(x+1, y+1)$ for all $x, y \in \mathbb{N}_0$. We then use the hypothesis (10) and the estimate in (12) to show that $G(x, y) \rightarrow 0$ as x or $y \rightarrow \infty$, and hence that G is identically zero for all non-negative integers x and y , so that (8) holds. \blacksquare

Theorem 2.4 Let $K(x, y)$ be as defined in (5), with $(\mathbf{a}(x))_{x=0}^{\infty} = ([A(x), B(x)]^T)_{x=0}^{\infty}$ a sequence of 2×1 real vectors such that

$$\sum_{x \geq 0} \|\mathbf{a}(x)\|^2 < \infty. \quad (14)$$

Suppose that there exists a sequence of 2×2 real matrices S_x such that $\mathbf{a}(x+1) = S_x \mathbf{a}(x)$ for all $x \in \mathbb{N}_0$ and that

$$C = \frac{S_y^T F S_x - F}{x - y} \quad (15)$$

is a constant matrix. Then C is symmetric. Suppose further that C has eigenvalues $\lambda \in \mathbb{R} \setminus \{0\}$ and 0, and let $[\alpha, \beta]^T$ be a real unit eigenvector corresponding to λ . Then

$$K(x, y) = -\operatorname{sgn}(\lambda) \sum_{k=0}^{\infty} \phi(x+k)\phi(y+k) \quad \text{for } x, y \in \mathbb{N}_0 \quad (x \neq y), \quad (16)$$

where

$$\phi(x) = |\lambda|^{1/2} (\alpha A(x) + \beta B(x)) \quad (17)$$

and $(\phi(x)) \in \ell^2$.

Proof. We set

$$C = \frac{S_y^T F S_x - F}{x - y} \quad (18)$$

where C is constant by hypothesis, so that we can exchange the roles of x and y , and find that $C^T = C$. We have, for $x \neq y$,

$$\begin{aligned} (\Delta_x S_y + \Delta_y)K(x, y) &= (S_x S_y - I) \frac{1}{x - y} \langle F \mathbf{a}(x), \mathbf{a}(y) \rangle \\ &= S_x \frac{1}{x - y - 1} \langle F \mathbf{a}(x), S_y \mathbf{a}(y) \rangle - \frac{1}{x - y} \langle F \mathbf{a}(x), \mathbf{a}(y) \rangle \\ &= \frac{1}{x - y} \langle F S_x \mathbf{a}(x), S_y \mathbf{a}(y) \rangle - \frac{1}{x - y} \langle F \mathbf{a}(x), \mathbf{a}(y) \rangle \\ &= \frac{1}{x - y} \langle (S_y^T F S_x - F) \mathbf{a}(x), \mathbf{a}(y) \rangle \\ &= \langle C \mathbf{a}(x), \mathbf{a}(y) \rangle. \end{aligned} \quad (19)$$

Since C is real and symmetric, and by hypothesis has eigenvalues $\lambda \neq 0$ and 0 , there exists a real orthogonal matrix U of unit eigenvectors such that

$$U^T C U = \begin{bmatrix} \lambda & 0 \\ 0 & 0 \end{bmatrix}. \quad (20)$$

We have

$$\begin{aligned} (\Delta_x S_y + \Delta_y)K(x, y) &= \langle C \mathbf{a}(x), \mathbf{a}(y) \rangle \\ &= \left\langle U \begin{bmatrix} \lambda & 0 \\ 0 & 0 \end{bmatrix} U^T \mathbf{a}(x), \mathbf{a}(y) \right\rangle \\ &= \lambda \left\langle \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} U^T \mathbf{a}(x), U^T \mathbf{a}(y) \right\rangle \\ &= \lambda \left\langle \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} U^T \mathbf{a}(x), \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} U^T \mathbf{a}(y) \right\rangle \\ &= \operatorname{sgn}(\lambda) \phi(x) \phi(y), \end{aligned} \quad (21)$$

where

$$\begin{bmatrix} \phi(x) \\ 0 \end{bmatrix} = \begin{bmatrix} |\lambda|^{1/2} & 0 \\ 0 & 0 \end{bmatrix} U^T \mathbf{a}(x). \quad (22)$$

Note that $(\phi(x)) \in \ell^2$ by the condition $\sum_{x \geq 0} \|\mathbf{a}(x)\|^2 < \infty$, since U is a constant matrix. It is also clear that $K(x, y) \rightarrow 0$ as x or $y \rightarrow \infty$, by the same condition on $\mathbf{a}(x)$. We now let $[\alpha, \beta]^T$ be a real unit eigenvector of C corresponding to λ , and the result follows by Lemma 2.3. \blacksquare

Corollary 2.5 *Let $K(x, y)$ be as defined in (5), with $(\mathbf{a}(x))_{x=0}^\infty = ([A(x), B(x)]^T)_{x=0}^\infty$ a sequence of 2×1 real vectors such that*

$$\sum_{x \geq 0} \|\mathbf{a}(x)\|^2 < \infty. \quad (23)$$

Suppose that $\mathbf{a}(x+1) = (Lx + M)\mathbf{a}(x)$ (for all $x \in \mathbb{N}_0$), where L and M are real constant 2×2 matrices that satisfy

$$\begin{cases} \det L = 0 \\ \det M = 1 \\ M^T FL \text{ is symmetric, and has eigenvalues } \lambda \in \mathbb{R} \setminus \{0\} \text{ and } 0. \end{cases}$$

Let $[\alpha, \beta]^T$ be a real unit eigenvector of $M^T FL$ corresponding to λ . Then

$$K(x, y) = -\operatorname{sgn}(\lambda) \sum_{k=0}^{\infty} \phi(x+k)\phi(y+k) \quad \text{for all } x, y \in \mathbb{N}_0 \quad (x \neq y), \quad (24)$$

where $\phi(x) = |\lambda|^{1/2} (\alpha A(x) + \beta B(x))$, and $(\phi(x)) \in \ell^2$.

Proof. We have $M^T FM = F \det M$ (indeed, this is true for any 2×2 matrix) and hence $M^T FM = F$. Likewise $L^T FL = 0$. Setting $S_x = Lx + M$ as in Theorem 2.4, we now have

$$\begin{aligned} \frac{S_y^T F S_x - F}{x - y} &= \frac{(Ly + M)^T F (Lx + M) - F}{x - y} \\ &= \frac{M^T FLx - (M^T FL)^T y}{x - y} \quad (\text{since } F^T = -F) \\ &= \frac{M^T FL(x - y)}{x - y} \quad (\text{since } M^T FL \text{ is symmetric by hypothesis}) \\ &= M^T FL. \end{aligned} \quad (25)$$

Hence $C = (S_y^T F S_x - F)/(x - y)$ is a constant matrix. Thus, together with the summability criterion on the sequence $(\mathbf{a}(x))$, the hypotheses of Theorem 2.4 are all satisfied, so we have the result. \blacksquare

Example 2.6

Let $J_\nu(z)$ be the Bessel functions of the first kind of order ν , and write $J_x = J_x(2\sqrt{\theta})$, where θ is a positive real parameter. The discrete Bessel kernel

$$J(x, y; \theta) = \sqrt{\theta} \frac{J_x J_{y+1} - J_y J_{x+1}}{x - y} \quad (26)$$

arises in the study of various discrete-variable random matrix models, as in [6] and [3]. Note that J_x is an entire function of x , so that $J(x, x; \theta)$ is well-defined via L'Hopital's rule. In the notation of Corollary 2.5, we take

$$\mathbf{a}(x) = \begin{bmatrix} \sqrt{\theta} J_x \\ J_{x+1} \end{bmatrix}. \quad (27)$$

The standard formula (see [8, p. 379])

$$e^{i2t \sin \theta} = J_0(2t) + 2 \sum_{m=1}^{\infty} J_{2m}(2t) \cos 2m\theta + 2i \sum_{m=1}^{\infty} J_{2m-1}(2t) \sin(2m-1)\theta \quad (28)$$

and Parseval's identity can be used to show that $J_0(2t)^2 + 2 \sum_{m=1}^{\infty} J_m(2t)^2 = 1$ for all real t , and hence that the sequence $(J_x)_{x=0}^{\infty}$ is square summable. Thus the condition $\sum_{x \geq 0} \|\mathbf{a}(x)\|^2 < \infty$ is satisfied.

The 3-term recurrence relation for the Bessel functions

$$J_{x+2}(2z) - \frac{x+1}{z} J_{x+1}(2z) + J_x(2z) = 0 \quad (29)$$

becomes

$$\mathbf{a}(x+1) = \begin{bmatrix} \sqrt{\theta}J_{x+1} \\ J_{x+2} \end{bmatrix} = \begin{bmatrix} 0 & \sqrt{\theta} \\ \frac{-1}{\sqrt{\theta}} & \frac{x+1}{\sqrt{\theta}} \end{bmatrix} \begin{bmatrix} \sqrt{\theta}J_x \\ J_{x+1} \end{bmatrix}, \quad (30)$$

and so we have $\mathbf{a}(x+1) = (Lx + M)\mathbf{a}(x)$, where

$$L = \begin{bmatrix} 0 & 0 \\ 0 & \frac{1}{\sqrt{\theta}} \end{bmatrix}$$

and

$$M = \begin{bmatrix} 0 & \sqrt{\theta} \\ \frac{-1}{\sqrt{\theta}} & \frac{1}{\sqrt{\theta}} \end{bmatrix}.$$

It is clear that these matrices satisfy $\det L = 0$ and $\det M = 1$, and we have

$$M^T F L = \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix},$$

so we pick the unit eigenvector $[\alpha, \beta]^T = [0, 1]^T$. Thus, the function $\phi(x)$ in Corollary 2.5 is $J_{x+1}(2\sqrt{\theta})$, and we recover a result of Borodin *et al* in [3]

$$J(x, y; \theta) = \sum_{k=0}^{\infty} J_{x+k+1}(2\sqrt{\theta}) J_{y+k+1}(2\sqrt{\theta}), \quad x, y \in \mathbb{N}_0, \quad (31)$$

without their use of asymptotic formulae for the Bessel functions.

The preceding results are identities of kernels for $x \neq y$. Evidently, the sum in the right-hand side of (16) makes sense for $x = y$, and hence gives one possible extension of the left-hand side to the case $x = y$. We use the extension to define an operator K with matrix given by $K(x, y)$.

Proposition 2.7 *Suppose that the vector $[A(x), B(x)]^T$ satisfies the conditions of Theorem 2.4, so that $K(x, y) = \sum_{k=0}^{\infty} \phi(x+k)\phi(y+k)$. Suppose also that $\sum_{n=0}^{\infty} n\phi(n)^2 < \infty$. Then the operator K represented by the matrix $[K(x, y)]_{x, y=0}^{\infty}$ is trace class and has trace:*

$$\text{trace } K = \sum_{x=0}^{\infty} (x+1)\phi(x)^2. \quad (32)$$

Proof. The summability condition on ϕ ensures that Γ_{ϕ} is Hilbert-Schmidt, which implies that $K = \Gamma_{\phi}^2$ is trace-class. We have

$$\text{trace } K = \sum_{x=0}^{\infty} K(x, x) = \sum_{x=0}^{\infty} \sum_{k=0}^{\infty} \phi(x+k)^2 \quad (33)$$

from which the result follows immediately. ■

Definition 2.8 *For a compact and self-adjoint operator W on a Hilbert space H , the spectral multiplicity function $\nu_W(\lambda) : \mathbb{R} \rightarrow \{0, 1, \dots\} \cup \{\infty\}$ is given by*

$$\nu_W(\lambda) = \dim\{x \in H : Wx = \lambda x\} \quad (\lambda \in \mathbb{R}). \quad (34)$$

We now give the consequences of a result of Peller, Megretski and Treil in [4] in the case of discrete integrable operators.

Proposition 2.9 *Suppose that Γ_ϕ and K are as in Proposition 2.7. Then Γ_ϕ and K are compact and self-adjoint, and*

- (i) $\nu_K(0) = 0$ or $\nu_K(0) = \infty$;
- (ii) for $\lambda > 0$, $\nu_K(\lambda) < \infty$ and $\nu_K(\lambda) = \nu_{\Gamma_\phi}(\sqrt{\lambda}) + \nu_{\Gamma_\phi}(-\sqrt{\lambda})$;
- (iii) if $\nu_K(\lambda)$ is even, then $\nu_{\Gamma_\phi}(\sqrt{\lambda}) = \nu_{\Gamma_\phi}(-\sqrt{\lambda})$;
- (iv) if $\nu_K(\lambda)$ is odd, then $|\nu_{\Gamma_\phi}(\sqrt{\lambda}) - \nu_{\Gamma_\phi}(-\sqrt{\lambda})| = 1$.

Proof. (i) follows from Beurling's theorem (see [5], page 15), while (ii) is elementary. Peller, Megretski and Treil show in [4] that for any compact and self-adjoint Hankel operator Γ , the spectral multiplicity function satisfies $|\nu_\Gamma(\lambda) - \nu_\Gamma(-\lambda)| \leq 1$. Using this, and (ii), statements (iii) and (iv) follow immediately. ■

Remark 2.10 *The Carleman operator $\Gamma : L^2(0, \infty) \rightarrow L^2(0, \infty)$ is given by*

$$\Gamma f(x) = \int_0^\infty \frac{1}{x+t} f(t) dt, \quad (35)$$

so Γ^2 has kernel of Tracy-Widom type

$$\Gamma^2 f(u) = \int_0^\infty \frac{\log u - \log t}{u-t} f(t) dt. \quad (36)$$

Carleman showed that Γ is a positive self-adjoint Hankel operator with continuous spectrum $[0, \pi]$ of multiplicity two (see [5, p. 442]), so the Tracy-Widom type operator Γ^2 has spectrum $[0, \pi^2]$, also of multiplicity two. This contrasts with (iii) and (iv) of Proposition 2.9.

3 Integrable operators on H^2

Let H^2 be the usual Hardy space on the unit circle \mathbb{T} , with orthonormal basis $\{1, z, z^2, \dots\}$, and let $R_+ : L^2 \rightarrow H^2$ and $R_- : L^2 \rightarrow L^2 \ominus H^2$ be the Riesz orthogonal projection operators. We let M_ϕ denote multiplication by ϕ , and define the Toeplitz operator on H^2 with symbol ϕ to be $T_\phi = R_+ M_\phi R_+$. Let $J : L^2 \rightarrow L^2$ be a flip operator, whose operation on a function $f \in H^2$ is $Jf(z) = \bar{z}f(\bar{z})$. Note that J maps H^2 onto $L^2 \ominus H^2$ (and vice versa) and that $J^2 = I$. The Hankel operator Γ_ϕ on H^2 with symbol $\phi \in L^\infty$ is then

$$\Gamma_\phi = JR_- M_\phi. \quad (37)$$

We let the integral operator W on $L^2(\mathbb{T})$ have kernel

$$W(e^{i\theta}, e^{i\phi}) = \frac{f(e^{i\theta})g(e^{i\phi}) - f(e^{i\phi})g(e^{i\theta})}{1 - e^{i(\theta-\phi)}}, \quad (38)$$

where W operates on a function $f \in L^2(\mathbb{T})$ in the usual way:

$$Wf(e^{i\theta}) = \frac{1}{2\pi} \int_{\mathbb{T}} W(e^{i\theta}, e^{i\phi}) f(e^{i\phi}) d\phi. \quad (39)$$

Lemma 3.1 *Suppose that $f, g \in L^\infty$ have $\bar{f} = g$. Then W defines a bounded and self-adjoint operator on L^2 . Further, $R_+WR_+ : H^2 \rightarrow H^2$ satisfies*

$$R_+WR_+ = \Gamma_f^*\Gamma_f - \Gamma_g^*\Gamma_g. \quad (40)$$

Moreover, when f is continuous, R_+WR_+ is compact.

Proof. The condition $\bar{f} = g$ gives immediately $W(e^{i\theta}, e^{i\phi}) = \overline{W(e^{i\phi}, e^{i\theta})}$, and so W is self-adjoint. It can easily be seen that the Riesz projection R_+ has distributional kernel $1/(1 - e^{i(\theta-\phi)})$, and so W decomposes as

$$W = M_g [M_f, R_+] - M_f [M_g, R_+], \quad (41)$$

where all the operators are bounded. A simple calculation now shows that

$$R_+WR_+ = (T_{gf} - T_gT_f) - (T_{fg} - T_fT_g), \quad (42)$$

and we apply the standard formulae $T_{hk} - T_hT_k = \Gamma_{h(\bar{z})}\Gamma_{k(z)}$ and $\Gamma_h^* = \Gamma_{\bar{h}(\bar{z})}$ (see [9, p. 253]) to get equation (40). The last statement follows by Hartman's theorem: the Hankel operators on the right-hand side of (40) are compact when f is continuous. ■

Remark 3.2 *We continue functions $f \in L^2$ to harmonic functions on \mathbb{D} by means of the Poisson kernel, as in [5, p. 718].*

Proposition 3.3 *Suppose $f = \bar{g} \in L^\infty$, where g is holomorphic inside \mathbb{D} . Then*

$$R_+WR_+ = \Gamma_f^*\Gamma_f. \quad (43)$$

Further, if R_+WR_+ has finite rank, then f is rational.

Proof. Take $f = \bar{g}$ in Lemma 3.1 to obtain the first part of the result. For the second part, note that

$$\text{Range}(R_+WR_+) = \text{Ker}(\Gamma_f^*\Gamma_f)^\perp = \text{Ker}(\Gamma_f)^\perp = \text{Range}(\Gamma_f^*) = \text{Range}(\Gamma_{\bar{f}(\bar{z})}), \quad (44)$$

and apply Kronecker's theorem: Γ_k has finite rank if and only if k is rational, so $\Gamma_{\bar{f}(\bar{z})}$ has finite rank if and only if $\bar{f}(\bar{z})$ is rational, which implies that f is rational. ■

Acknowledgements

I would like to thank my supervisor, Gordon Blower for his patience in many discussions. Thanks are also due to Steve Power for helpful suggestions. I am funded by the EPSRC's Doctoral Training Account scheme.

References

- [1] C.A Tracy and H. Widom, Level-spacing distributions and the Airy kernel, Comm. Math. Phys. 159 (1994), 151-174

- [2] G Blower Operators associated with Soft and Hard Spectral Edges from Unitary Ensembles, J. Math. Anal. Appl. 2007.
- [3] A. Borodin, A. Okounkov, G. Olshanski, Asymptotics of Plancherel measures for symmetric groups, J. Amer. Math. Soc. 13 (2000) 481-515.
- [4] A.V. Megretskii, V.V. Peller, and S.R. Treil, The inverse spectral problem for self-adjoint Hankel operators, Acta Math. 174 (1995), no. 2, 241-309
- [5] V.V. Peller, Hankel Operators and Their Applications, Springer, New York, 2003.
- [6] K. Johansson, Discrete Orthogonal Polynomial ensembles and the Plancherel measure, Ann. Math. (2) 153 (2001), no. 2, 259-296.
- [7] S.C. Power, Hankel Operators on Hilbert Space, Research Notes in Math., 64, Pitman Adv. Publ. Progr., Boston-London-Melbourne, 1982.
- [8] E.T Whittaker and G.N. Watson, A Course of Modern Analysis, Cambridge University Press, 4th ed. 1963
- [9] N.K. Nikolski, Operators, Functions and Systems: an easy reading, vol. 2 American Mathematical Society, Providence, R.I., 2002.